# Sudden Acceleration of a Laminar Boundary Layer by a Moving Belt

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**An incompressible, laminar boundary layer of the Falkner-Skan type encountering a sudden change in boundary conditions is analyzed by using a transformation into a new coordinate system originating at the encounter location. The equations so obtained are solved far away from, as well as close to, the encounter plane. Results show that at the same downstream position the effect of a slowly moving belt tends to penetrate deeper into the boundary layer than the effect of a very rapidly moving belt. They also show that separation of the boundary layer can be shifted infinitely far downstream, and that sufficiently rapid movement of the belt can produce in a short distance a substantial acceleration of the flow leading to over-all frictional thrust instead of drag.**

## Nomenclature

$A$ = scaling constant for freestream velocity
$a$ = translational constant in transformation of $\eta$
$a_n$ = constant coefficients in expansion of $(1 - \zeta)^{1/2}$
$B^2$ = belt speed parameter $[= W(1 + m)/2w]$
$b$ = stretching constant in transformation of $\eta$
$b_n$ = constant coefficients in expansion of $(1 - \theta^2)^m$
$C_f$ = friction coefficient $[=(2\nu/U^2)(\partial\bar{u}/\partial y)_{y=0}]$
$c_n$ = constants determining $T_n(t)$ $(n = 0, 1, 2 \ldots)$
$F(\eta^0)$ = Falkner-Skan stream function
$f(\eta, \theta \text{ or } \zeta)$ = normalized stream function of the perturbation layer
$f_n(\eta)$ = functional coefficients in the expansion of $f$ $(n = 0, 1, 2 \ldots)$
$G(\eta)$ = normalized stream function for flow over continuously moving surface
$I_{1,n}, I_{2,n}$ = integrals required in computation of $\delta_2$
$k$ = permutation index
$L$ = distance from leading edge to the moving belt (also location of the encounter plane)
$m$ = exponent of $x$ in the Falkner-Skan solutions
$n$ = exponent of $\theta$, $\zeta$, or $z$ in the expansions of $f$, $\phi$, $S$, or $T$ $(n = 0, 1, 2 \ldots)$
$p$ = static pressure
$Re$ = local Reynolds number $(= Ux/\nu)$
$S(\eta, \theta \text{ or } \zeta)$ = transformed Falkner-Skan function
$S_n(\eta)$ = functional coefficients in the expansion of $S$ $(n = 0, 1, 2 \ldots)$
$T(t, z)$ = transformed Falkner-Skan function
$T_n(t)$ = functional coefficients in the expansion of $T$ $(n = 0, 1, 2 \ldots)$
$t$ = transformed normal coordinate $(= \eta/Bz)$
$U$ = streamwise velocity outside the boundary layer
$u$ = streamwise velocity in the boundary layer
$v$ = velocity in the direction normal to the belt
$W$ = value of $U$ at $x = L$
$w$ = belt speed
$x$ = streamwise coordinate measured from leading edge
$y$ = coordinate normal to the belt
$z$ = transformed normalized streamwise coordinate $(= \zeta^{-m/2})$
$\delta_1$ = displacement thickness
$\delta_2$ = momentum thickness
$\zeta$ = modified normalized streamwise coordinate
$\eta$ = normalized coordinate perpendicular to the belt
$\eta^0$ = Falkner-Skan similarity variable
$\theta$ = normalized streamwise coordinate $[= (\xi/x)^{1/2}]$
$\mu$ = viscosity
$\nu$ = kinematic viscosity
$\xi$ = streamwise coordinate measured from the encounter plane
$\rho$ = density
$\phi(t, z)$ = transformed, normalized perturbation stream function
$\phi_n(t)$ = functional coefficients in the expansion of $\phi$ $(n = 0, 1, 2 \ldots)$
$\psi$ = perturbation stream function

***Subscripts*†**

$\infty$ = limiting value at infinity
$*$ = transformed value

***Superscripts***

0 = undisturbed flow
$-$ = bar over value denotes sum of undisturbed and perturbing flows
$'$ = denotes total differentiation

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## I. Introduction

IN 1904 Prandtl demonstrated[1] that a rotating cylinder could prevent the separation of the boundary layer. He correctly explained[2] that this prevention of separation occurred because the cylinder, by dragging the flow adjacent to its surface with a velocity greater than that of the freestream, was able to overcome the effect of the rising pressure which caused the separation-producing retardation. As a result of Prandtl's explanation, other devices for acceleration of the boundary layer were devised. One of these was a so-called Handley-Page (also known as Lachman) flap,[2] which created a streamwise-directed accelerating jet to serve the same purpose as a moving surface. Another proposed device[3] involved the replacement of the top surface of the airfoil by a moving band. Although the Handley-Page and Lachman flaps had many practical applications, the use of moving belts or rotating cylinders for boundary layer acceleration has not

† Coordinate subscripts indicate partial differentiation.

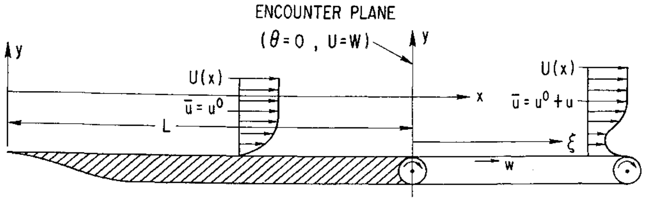


**Fig. 1 Schematic representation of velocities and coordinates.**

been practiced. Recently, however, a moving belt was successfully used to eliminate a thick boundary layer in a wind tunnel[4] and a rotating cylinder was proposed by V/STOL aircraft to achieve unseparated flow over a highly cambered airfoil.[5]

The idea of the sudden acceleration of the boundary layer by a moving boundary is thus not only practical but also quite well known. It is surprising, therefore, that this problem has received so little attention from the analytical standpoint. To the author's knowledge only two papers exist that deal with this type of boundary-layer acceleration. The first of these[6] proposes a semiempirical method, based on experimental data, which permits the calculation of the effect of a tangential jet on a turbulent boundary layer subjected to an intense adverse pressure gradient. Since the singly injected jet is continuously attenuated by the wall friction and thus exhibits a definite maximum velocity (e.g., vanishing normal derivative) in the vicinity of the wall, this problem is somewhat different from that of the boundary-layer acceleration by a steadily moving wall.

The second paper[7] utilizes series expansions to obtain inner and outer solutions of nonsimilar, laminar, boundary layers with very general boundary conditions. Because of this generality, the authors were able to study the special case of a circular cylinder which, in the vicinity of the separation region, had a varying wall velocity in the streamwise direction. This wall velocity, at first increasing linearly and then remaining constant, was supposed to simulate a continuously distributed and suitably dense mass injection through the wall. This problem, then, though slightly different, is closely related to the case in which the wall itself is moving in the streamwise direction.

It is regrettable, therefore, that the authors of Ref. 7 did not see fit to enclose a detailed description of the changes that occurred as a result of the streamwise wall velocity, though they did indicate a reduction of friction and a slight downstream shift of the separation point. Unfortunately, there is serious doubt that the authors applied a valid criterion for separation since the vanishing of the friction coefficient seems to have been taken as an indicator of the onset of separation and no computations were carried beyond this point. It is easy to see that the vanishing of the friction coefficient cannot be used to indicate flow separation over a moving wall. For example, when the wall is moving at the same speed as the external, potential flow, and since there is no relative motion between the wall and the fluid, the wall friction may be expected to vanish, and yet the flow will remain attached. What is then the proper indication of separation over a moving surface? Moore,‡ who studied this problem, came to the conclusion that the simultaneous vanishing of the streamwise velocity and its derivative with respect to the direction normal to the wall, collocated somewhere in the boundary layer, is the proper criterion for the formation of a separation bubble. Clearly then, to determine the separation one needs to know the entire streamwise velocity profile. Such information is not available in Ref. 7. For these reasons the general results of that reference are questionable and do not contribute to the detailed understanding of the changes arising from the acceleration of the boundary layer by the moving belt.

The purpose of the present paper is to fill this void and analytically study the process of sudden acceleration of a laminar boundary layer by a moving belt. Notice that this is fundamentally different from those problems in which the boundary layer is formed over a continuously moving surface as studied in Refs. 8–13.§ This is so because, starting at the plane where the moving boundary is first encountered, the sudden acceleration produces modifications to an already existing and well-developed boundary layer which originated upstream at some leading edge; whereas the boundary layer over a continuously moving surface develops without any sudden changes, all of it starting at one common origin. Thus the sudden acceleration, if sufficiently strong, must produce dissimilar velocity profiles whereas the boundary layer over a continuously moving surface can have the same normalized velocity profile everywhere. This difference is common to a class of boundary-layer problems involving a sudden change in boundary conditions. In the past, these problems, though important, were mainly neglected (except for discontinuous temperature distribution which received a fair degree of attention), and it is hoped that this paper will stimulate additional studies in this direction.

It is easy to see that eventually, far downstream from the plane where the boundary layer first encounters the moving belt, the effect of the sudden acceleration has a chance to propagate over the complete thickness of the original layer so that in this region, under suitable conditions, the boundary layer should tend to regain its similarity behavior. Intuitively, such a fully adjusted boundary layer will correspond to a flow which has developed over a continuously moving surface. Thus, the solutions of Refs. 9–13¶ should be applicable there.

The present work is divided into four parts. The first three parts concern themselves with the development of the basic equations and their solutions both far away and close to the plane of encounter between the undisturbed boundary layer and the moving belt. Some of these solutions were obtained by high-speed electronic computers. The fourth part of the present paper utilizes these mathematical results in describing the effects of the moving belt on the physical characteristics of the boundary layer.

## II. General Equations

Assuming that all velocities (see Fig. 1) are taken relative to the stationary portion of the surface over which the boundary layer is flowing, and using the usual approximations applicable to thin, incompressible flow layer, one states the conservation of mass and momentum as

$$\bar{u}_x + \bar{v}_y = \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + (1/\rho)\bar{p}_x - \nu\bar{u}_{yy} = \bar{p}_y = 0 \quad (1)$$

with the boundary conditions: $\bar{u}(x, \infty) = U(x)$; $\bar{u}(0 \leq x < L, 0) = 0$; $\bar{u}(L \leq x \leq \infty, 0) = w$; $\bar{v}(x, 0) = 0$.

Next, the boundary-layer velocities are assumed to consist of 2 parts—those undisturbed and those caused by the belt movement: $\bar{u}(x, y) = u^0(x, y) + u(\xi, y)$; $\bar{v}(x, y) = v^0(x, y) + v(\xi, y)$ where the coordinate $\xi$ is defined by the translation $\xi = x - L$ so that if for all $\xi < 0$ $u$ and $v$ are understood to vanish, then the boundary conditions may be conveniently separated into: $u^0(x, 0) \equiv v^0(x, 0) = u(\xi, \infty) = v(\xi, 0) = 0$;

‡ According to Moore,[8] Sears and Rott have also arrived independently at the same result.

§ Although Refs. 8–11 pertain to unsteady boundary layers, in the coordinates moving with the speed of either the advancing wave or the separation point, the boundary layer is steady and sees the wall as a continuously moving surface.

¶ The solution of Ref. 8 is obtained over a "wall" whose speed along its length always varies like the external velocity. This assumption makes the solution of that reference inapplicable to the present problem.

$u^0(x, \infty) = U$; $u(\xi, 0) = w$, and Eq. (1) may be written as

$$u^0_x + v^0_y = u^0u^0_x + v^0u^0_y - UU_x - \nu u^0_{yy} = 0 \quad (2a)$$

$$u_x + v_y = u^0u_x + uu^0_x + uu_x + v^0u_y + vu^0_y + vu_y - \nu u_{yy} = 0 \quad (2b)$$

The solution of Eq. (2a) for arbitrary $U(x)$ is well known.[14] In principle, therefore, it would be possible to use such a solution in what follows. This, however, would deny us the ability to discern easily between the effects of the arbitrary pressure gradient and those due to the belt movement. For this reason we shall restrict our considerations to the Falkner-Skan similarity solutions[15] so that the departure from similarity can be an indication of the adjusting process to the suddenly changed boundary condition. These solutions require that $U = Ax^m$ and then $u^0$ and $v^0$ are given by $u^0 = UF'$; $v^0 = \{U\nu/[2(1+m)x]\}^{1/2}[(1-m)\eta^0F' - (1+m)F]$ where $F$ is a function of $\eta^0$ only and satisfies the equation

$$F''' + FF'' + [2m/(1+m)](1 - F'^2) = 0$$

with

$$F(0) = F'(0) = 0; \quad F'(\infty) = 1 \quad (3a)$$

and $\eta^0 \equiv y[(1+m)U/(2\nu x)]^{1/2}$ is the similarity variable.

The numerical values of function $F$, and its derivatives for various values of $m$, were computed by Hartree.[16] Therefore, considering $u^0$ and $v^0$ as known, one may next proceed to the solution of Eq. (2b). Specifically, setting $\theta \equiv (\xi/x)^{1/2} = [1 - (L/x)]^{1/2}$ one may use it in the transformation $\eta^0 = \eta\theta$ so that $\eta$ is the contracted pseudo-similarity parameter of a new boundary layer, starting to grow at the encounter plane and whose nondimensional stream function $f(\eta, \theta)$ can be defined by $\psi \equiv wy[1 - (f/\eta)]$. Thus the velocity components which satisfy the conservation of mass identically are given by $u = \psi_x$ and $v = -\psi_y$ or

$$u = w(1 - f_\eta)$$
$$v = wy[(m\theta^2 - 1)(\eta f_\eta - f) + \theta(1 - \theta^2)f_\theta]/(2\theta^2 x\,\eta) \quad (4)$$

Next it is necessary to transform the function $F$ into the $\eta$, $\theta$ coordinate system. As shown in Ref. 17, this may be accomplished by using the transformation $F(\eta^0) = \theta S(\eta, \theta)$ so that $F' = S_\eta$, $F'' = \theta^{-1}S_{\eta\eta}$ and $F''' = \theta^{-2}S_{\eta\eta\eta}$. The function $S$ must thus satisfy

$$S_{\eta\eta\eta} + \theta^2\left[SS_{\eta\eta} + \frac{2m}{1+m}(1 - S_\eta^2)\right] = \eta S_\eta - \theta S_\theta - S = 0 \quad (3b)$$

with $S = S_\eta = 0$ at either $\eta = 0$ or $\theta = 0$, $S_\theta = 0$ at $\eta = 0$ and $S_\eta = 1$ for $\eta \rightarrow \infty$.

Using these expressions in the statement of the conservation of momentum of Eq. (2b), one obtains after some manipulations

$$(1+m)f_{\eta\eta\eta} + [(1-\theta^2)\eta S_\eta + (1+m)\theta^2 S]f_{\eta\eta} - (1-\theta^2)[\theta S_\eta f_{\eta\theta} + S_{\eta\eta}(\eta f_\eta - \theta f_\theta)] + 2m\theta^2 S_\eta(1 - f_\eta) + (1 - m\theta^2)S_{\eta\eta}f - (1-m)\theta^2\eta S_{\eta\eta} + w/W\,(1-\theta^2)^m\{(1 - m\theta^2)(\eta - f)f_{\eta\eta} - \theta(1-\theta^2)[f_\theta f_{\eta\eta} + (1 - f_\eta)f_{\eta\theta}]\} = 0 \quad (5a)$$

where at $\eta = 0$: $f = f_\eta = f_\theta = 0$ and for $\eta \rightarrow \infty$: $f_\eta = 1$. The velocity $W$, which appears in this equation, is defined by $W \equiv U(L)$ so that $U = W(1 - \theta^2)^{-m}$.

The solution of Eq. (5a) is difficult because it is rather lengthy, nonlinear, partial differential equation, whose coefficients (e.g., functions $S$, $S_\eta$, etc.) are generally known in numerical form only. For these reasons all subsequent considerations will now be limited to regions where the form of this equation is amenable to further simplifications. These regions lie far away and also very near to the plane in which the unperturbed boundary layer first encounters the moving belt.

## III. Perturbed Flow Far Downstream from the Encounter Plane

### A. Applicable Equation

By restricting the solution to values of $\theta$ approaching unity, one may reduce Eq. (5a) to a more easily solvable form. As may be ascertained from the definition of $\theta$, this restriction means that either the flow takes place very far downstream from the encounter plane (e.g., $x \gg L$), or that the encounter plane coincides with the leading edge plane, and the flow takes place over a continuously moving surface. In this region it is appropriate to define a new variable $\zeta \equiv 1 - \theta^2 = L/x$, and study the flow corresponding to values of $\zeta$ approaching zero. Equation (5a) may thus be given the following form [all omitted terms are given in Eq. (9b) of Ref. 17.]:

$$(1+m)f_{\eta\eta\eta} + \ldots + \zeta(2S_\eta f_{\eta\zeta} + \ldots) - \zeta^2(S_{\eta\eta}f_\zeta + \ldots) = (w/W)\zeta^m[ff_{\eta\eta} + \ldots + \zeta(2f_\eta f_{\eta\zeta} + \ldots)\zeta^2(2f_\zeta f_{\eta\eta} + \ldots)] \quad (5b)$$

Next when $m \geq 0$ and is an integer, Eq. (5b) may be changed into a system of total differential equations by introducing the expansions**

$$f = \sum_{n=0}^{\infty}\zeta^n f_n(\eta); \quad S = \sum_{n=0}^{\infty}\zeta^n S_n(\eta) \quad (6)$$

This leads to a set of recurring equations for $f_n(\eta)$ [see Eqs. (9c) of Ref. 17] whose boundary conditions are $f_n(0) = f_n'(0) = f_{n\neq 0}'(\infty) = 0$; $f_0'(\infty) = 1$. Furthermore, the functions $S_n$, which also occur in these equations, are shown in Appendix A of Ref. 17 to be easily derivable from $S_0 = F(\eta)$.

### B. Slow Belt Speed

When the belt moves very slowly relative to the external flow, the ratio $w/W$ may be set to zero. This disposes of all nonlinear terms and is completely equivalent to the neglection of $uu_x$ and $vu_y$ in Eq. (2b). Notice that since all nonlinear terms in Eq. (5b) are multiplied by $\zeta^m$, therefore large, positive values of $m$ will have the same effect as the neglection due to the linearization. Thus the solution for slow belt speed†† should be always applicable to cases where the external flow is strongly accelerating, even when $w/W$ is not necessarily small. This occurs because for such a flow and large $x/L$ the external velocity rapidly increases to very large values, thus making the constant belt speed (and therefore also its perturbation) less and less significant.

When the aforementioned linearization is carried out in the recurring equations of Ref. 17, then the equation for $f_0$ becomes

$$f_0''' + S_0f_0'' + \frac{2m}{1+m}S_0'(1 - f_0') - \frac{1-m}{1+m}S_0''(\eta - f_0) = 0$$

with

$$f_0(0) = f_0'(0) = 0 \text{ and } f_0'(\infty) = 1 \quad (7)$$

This equation was solved numerically, and the values of $f_0'$ for $m = 0, \frac{1}{3}, \frac{2}{3}, 1$, and $\infty$ are given in Table 1 of Ref. 17. Negative values of $m$ were not considered because the inspection of Eq. (5b) reveals that for such values of $m$ the role of

** The same expansions can also be used when $m$ is not an integer but $w/W$ is small.

†† Using, of course, the proper value of $m$.

the nonlinear terms increases in importance and their neglection is not valid.

### C. Accelerating External Flow

To carry out the solution of the nonlinear equations one must restrict $m$ to integer values. For this reason our discussion will be limited to the stagnation flow for which $m = 1$. Substituting this value into the recurring equations of Ref. 17 and taking $n = 0$, one obtains

$$f_0''' + S_0 f_0'' + S_0'(1 - f_0') = 0$$

with

$$f_0(0) = f_0'(0) = 0 \text{ and } f_0'(\infty) = 1 \tag{8}$$

Similarly, the higher successive values of $n$ give the equations for $f_1, f_2$, etc.

Notice that although we have made no assumptions on the magnitude of $w/W$, Eq. (8) is linear and could have been obtained by setting $m = 1$ in Eq. (7). This linear character, in fact, remains true also for $f_1$. Here then is evidence supporting our previous assertion that the linearized solutions will be applicable to accelerating flows, regardless of the magnitude of $w/W$. The solutions for $f_0, f_1$, and $f_2$ were computed numerically, and the values of $f_0', f_1'$, and $f_2'$ so obtained are given in Tables 2a, 3a, and 4a of Ref. 17.

### D. Constant External Velocity

When $m = 0$ and $n = 0$ the recurring equations of Ref. 17 give

$$f_0''' + S_0 f_0'' + (\eta - f_0)[(w/W)f_0'' - S_0''] = 0$$

with

$$f_0(0) = f_0'(0) = 0 \text{ and } f_0'(\infty) = 1 \tag{9}$$

For $w = W$, the solution of Eq. (9) may be obtained by inspection and is obviously $f_0 = S_0$. Thus $\bar{u} = w = W$ everywhere in the boundary layer (and not just at the boundaries) or, in other words, when the belt is moving with the speed of the freestream the boundary layer eventually tends to disappear. That this is in fact so, is not only intuitively plausible, but is also confirmed by the data of Ref. 4, where for $w = 1.05\ W$ a thick boundary layer essentially vanishes.

When $w$ is different from $W$, one may introduce $G(\eta)$ such that $\bar{u} = WG'$ by setting $f_0 = \eta + (W/w)(S_0 - G)$ which, when substituted into Eq. (9), yields the well-known Blasius equation $G''' + GG'' = 0$, whose boundary conditions, however, are $G(0) = 0$, $G'(0) = (w/W)$, and $G'(\infty) = 1$. This equation and its boundary conditions are identical with those obtainable for a boundary layer developing over a continuously moving surface. Their solution for various values of $(w/W)$ was first given by Mirels.[9,10] Using therefore his solutions in equations for higher values of $n$, one can find the functions necessary for the description of the flow when $\zeta \neq 0$. The results of such computations for $f_0', f_1'$, and $f_2'$ with $w/W = 0.1$, 1, and 10 are given in Tables 2a, 3a, and 4a of Ref. 17.

### E. Decelerating External Flow

When $m$ is negative, one needs to multiply Eq. (5b) by $\zeta^{-m}$. The solution of the resulting equation for $f_0$ is $f_0 = \eta$. This solution however indicates that the $u$ flow vanishes everywhere, except perhaps at $\eta = 0$, where the boundary condition $f_0'(0) = 0$ is obviously violated. This inability to satisfy all boundary conditions occurs because as $\zeta$ tends to zero, the order of the differential equation is reduced, indicating a singular perturbation problem. Such a problem is fully discussed in Ref. 18. To deal with it, one uses the transformations

$$\eta = Bzt; \quad \zeta = (z)^{-2/m}; \quad S(\eta, \zeta) = BzT(t, z)$$
$$f(\eta, \zeta) = Bz[t - \phi(t, z)] \tag{10}$$

with $B^2 = \frac{1}{2}(1 + m)(W/w)$. These change Eq. (5b) into

$$2\phi_{ttt} + \phi_{tt} - mz(\phi_{zt}\phi_t - \phi_z\phi_{tt}) + (W/w)z^2\{[(1 + m)T + mtT_t]\phi_{tt} + m(\tfrac{1}{2}tT_{tt} - 2T_t)\phi_t + (1 - \tfrac{3}{2}m)T_{tt}\phi - mz(T_t\phi_{zt} + \tfrac{1}{2}T_{tt}\phi_z)\} = 0[(z)^{-2/m}] \tag{11}$$

It is appropriate therefore to expand the functions $\phi$ and $T$:

$$\phi(t, z) = \sum_{n=0}^{\infty} z^n\phi_n(t); \quad T(t, z) = \sum_{n=0}^{\infty} z^nT_n(t) \tag{12}$$

where the function $T$ must, of course, satisfy the transformed form of Eq. (3). This is carried out in Ref. 17 where it is pointed out that terms of $O[(z)^{-2/m}]$ may comfortably be neglected, upon which the solutions for $T_n$ are of the form $T_n = c_nt^{(1+n)}$ with $c_0 = 0$, $c_1 = \frac{1}{2}BF''(0)$ and $c_2 = -(2m/6) \times (W/w)$. This form of $T_n$ could, of course, have been expected, by realizing that the singular nature of the problem restricts the considerations to the immediate neighborhood of the moving belt.

Substituting Eq. (12) into Eq. (11) and again neglecting terms of $O[(z)^{-2/m}]$ one obtains [see Eq. (16) of Ref. 17] the recurring relations for $\phi_n$ with the boundary conditions $\phi_n(0) = \phi_{n\neq 0}'(0) = \phi_n'(\infty) = 0$ and $\phi_0'(0) = 1$. For $n = 0$ these relations reduce to the Blasius equation

$$2\phi_0''' + \phi_0\phi_0'' = 0 \tag{13}$$

which, using the present boundary conditions, has been solved by Sakiadis (Ref. 12). For separating flow both $c_0$ and $c_1$ are zero and this, together with the boundary conditions, leads to $\phi_1 = \phi_2 = \phi_3 = 0$. Thus for separating flow, the Sakiadis solution is applicable up to terms of $O(z^4)$. But the unperturbed flow vanishes with the freestream velocity $U$ like $\zeta^{-m}$ (or $z^2$) so that it is clear from the preceding discussion that to this order of approximation no additional computations are necessary.

Notice that a singular solution identical to $\phi_0$ has been suggested by Moore,[8] who studied the transient movement of the separation point by relating it to the steady flow with a fixed separation point, but over a continuous, moving surface. The need for a singular solution of Ref. 8 arises when the surface is moving upstream, so that as separation is approached, the boundary layer is decelerated purely by the action of the pressure gradient. Since this is the mechanism which is also responsible for the vanishing of the boundary-layer flow in the present case, the equivalence of the two singular solutions is not surprising.

## IV. Perturbed Flow Close to the Encounter Plane

For small values of $\theta$, that is, close to the encounter plane, when the perturbing boundary layer is thin relative to the original layer, one may expand all quantities in powers of $\theta$:

$$f = \sum_{n=0}^{\infty} \theta^n f_n(\eta); \quad S = \sum_{n=0}^{\infty} \theta^n S_n(\eta); \quad (1 - \theta^2)^m = \sum_{n=0}^{\infty} \theta^n b_n \tag{14}$$

Here again the solutions for $S_n$ are given in Appendix A of Ref. 17 and the coefficients $b_n$ are easily obtainable and are, of course, equal to zero for all odd values of $n$.

Substituting Eqs. (14) into Eq. (5a) leads to the recurring relations for $f_n$. These equations are given in Ref. 17 [Eq. (19)] where it is shown also that $S_0 = 0$. For $n = 0$, one thus obtains

$$2B^2f_0''' + (\eta - f_0)f_0'' = 0 \tag{15}$$

which, as shown in Ref. 17, can be transformed by

$$\eta = B(2)^{1/2}\left[a - \eta_*\left(\frac{b}{2}\right)^{1/2}\right]; \quad f_0(\eta) = \eta + \frac{B}{(b)^{1/2}}f_*(\eta_*) \tag{16}$$

into the Blasius relation in $f_*$, whose solution, however, is subject to certain very special boundary conditions. Fortunately, this equation, together with its boundary conditions, has been solved by Lock (Ref. 19) who studied the velocity distribution in the boundary layer between parallel streams. The function $f_0$ may thus be considered known. Next, using the recurring equations of Ref. 17 one may numerically compute all the other $f_n$ functions. The values so obtained of $f_1'$ and $f_2'$ are tabulated in Tables 3b and 4b of Ref. 17.

## V. Effects of the Moving Belt on Boundary-Layer Characteristics

### A. Changes of the Velocity Distribution

To illustrate the effect of the belt on the velocity distribution in the boundary layer, three specific examples were computed. However, before proceeding with these, certain general observations, which follow out of the nature of the mathematical solutions, are in order. Thus, concentrating on the flow close to the encounter plane, we note from Eq. (16) that the larger the value of $B$, the higher the value of $\eta$ at which $f_0'$ will tend to reach its freestream limiting value. Furthermore, although this transformation has been used in connection with $f_0'$ only, one may ascertain from Tables 3b and 4b of Ref. 17 that this tendency also exists for $f_1'$ and $f_2'$. Since $\eta^0 = \eta\theta$, this means that for a fixed distance downstream from the encounter plane (e.g., fixed $\theta$), the lower the ratio of the belt to the external flow velocity and the higher the acceleration of the external flow, the deeper the penetration of the effect of the moving belt into the original layer. Apparently, the original boundary layer, particularly when accelerating, has little difficulty in adjusting to small changes in boundary conditions. Conversely, this also means that, close to the encounter plane, a very rapidly moving belt will tend to affect only that flow which is immediately adjacent to its surface, leaving the outer portions of the boundary layer essentially undisturbed. This is well illustrated in the first example shown in Fig. 2 which concerns an accelerating boundary layer ($m = 1$) encountering a belt moving ten times faster than the freestream. As may be seen from Fig. 2, where the ratio of the boundary layer to the freestream velocity is plotted against the original similarity parameter $\eta^0$, for small values of $\theta$ the effect of the belt is limited to values of $\eta^0$ which are smaller than unity. Of course, far from the encounter plane the influence of the moving belt extends much deeper into the boundary layer; but there the accelerating outer flow nullifies the moving belt, and the flow eventually tends to regain its original similarity profile.

The second example shown in Fig. 3 illustrates how in the absence of pressure gradient a belt moving with the freestream velocity eliminates the boundary layer. Notice that this elimination occurs rather rapidly and thus, for instance,

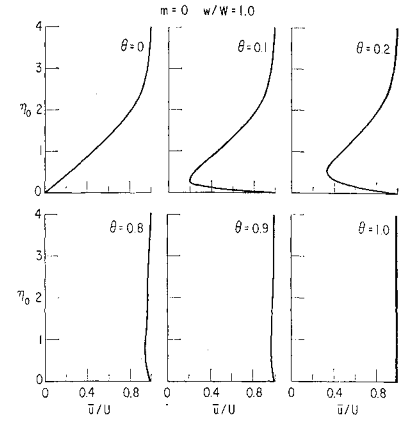


**Fig. 3 Elimination of a boundary layer by a belt moving with the external flow.**

at $\theta = 0.8$ the boundary-layer velocity is nowhere smaller by more than 8% than the freestream velocity.

And finally the third example shown in Fig. 4 demonstrates how even a very slowly moving belt ($w = 0.1\,W$) is capable of shifting the separation point infinitely far downstream [as may be easily ascertained from Eq. (10), $z$ corresponds to values of $\theta$ very nearly equal to unity]. In fact, since for this case $m$ is negative so that the external flow is continuously decelerating, one can see from Fig. 4 that the separation occurs only because at infinity the external flow (and with it the boundary layer) ceases to exist.

### B. Effect on the Friction Coefficient

The effect of the moving belt on the friction coefficient far from the encounter plane is given (for $m \geq 0$) by

$$C_f(Re)^{1/2} = [2(1+m)]^{1/2}\{F'' - (w/W)\zeta^m[f_0'' + \zeta(f_1'' + \tfrac{1}{2}f_0'') + \zeta^2(f_2'' + \tfrac{1}{2}f_1'' + \tfrac{3}{8}f_0'')]\}_{\eta=\eta^\circ=0} \quad (17a)$$

and close to the encounter plane (for all $m$) by

$$C_f(Re)^{1/2} = [2(1+m)]^{1/2}\ \{F'' - (w/W)[(1/\theta)f_0'' + f_1'' + \theta(f_2'' - mf_0'')]\}_{\eta=\eta^\circ=0} \quad (17b)$$

Equations (17a) and (17b) are plotted in Fig. 5. As can be seen from this figure and Eq. (17b), at the encounter plane the friction coefficient tends to negative infinity like $1/\theta$ [e.g., like $(x/\xi)^{1/2}$]; and this, except for the change in the direction of the force, is completely analogous to the usual boundary-layer behavior at the leading edge. Downstream from the encounter plane the friction coefficient rises to values which, in general, are lower than those obtainable over an immovable wall. This becomes especially clear when, for example, the boundary layer with constant external velocity and infinitely far from the encounter plane is considered. As previously discussed, the solutions of such a flow were given by Mirels.[9,10] Using his correlations in Eq. (17a), one may write the limiting ratio of the friction coefficients with and without the moving belt (which, incidentally, for this special case is also equivalent

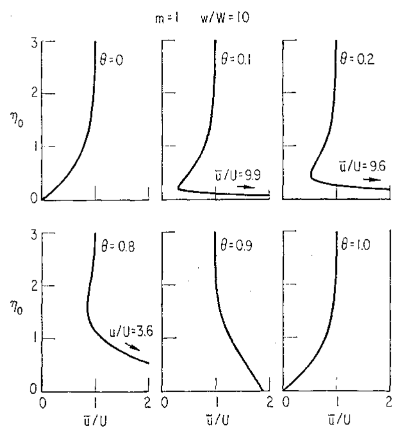


**Fig. 2 Effect of rapidly moving belt on the velocity profile of an accelerating flow.**

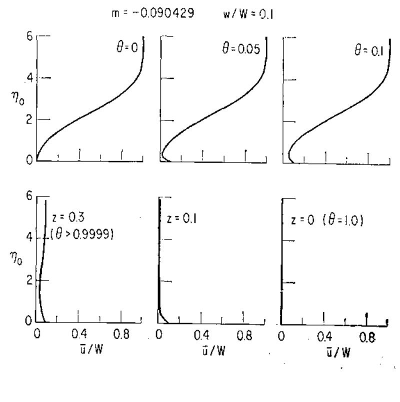


**Fig. 4 Delay of separation by a slowly moving belt.**

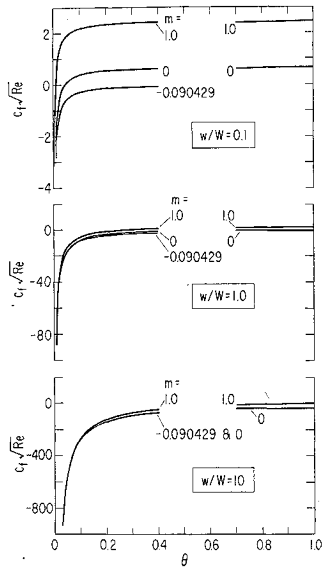


**Fig. 5 Local "friction" coefficients.**

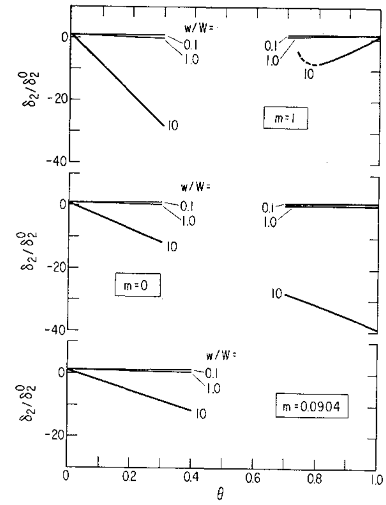


**Fig. 6 Ratio of the momentum thicknesses.**

to the ratio of the momentum thicknesses) as

$$C_f/C_f^0 = \delta_2/\delta_2^0 = \frac{G''(0)}{F''(0)} = \begin{cases} \left(1 - \frac{w}{W}\right)\left(1 + 1.887\,\frac{w}{W}\right)^{1/2} & 0 \leq \frac{w}{W} \leq 1 \\ 1.04\left(1 - \frac{w}{W}\right)\left(1 + 1.665\,\frac{w}{W}\right)^{1/2} & 1 \leq \frac{w}{W} \leq 6‡‡ \end{cases} \tag{18}$$

Equation (18) indicates that the limiting values of the friction coefficient with the belt moving are always smaller than those for the stationary wall. Consequently, it follows from Fig. 5 that in the absence of pressure gradient the moving belt reduces everywhere the value of the friction coefficient.

Similar behavior occurs also when the outer flow is accelerating, except that in this case the effect of the belt disappears infinitely far away from the encounter plane so that the friction coefficient eventually rises to the same values as those obtainable for a stationary wall.

For decelerating external flows and far from the encounter plane the values of the conventionally defined friction coefficient tend to infinity because of the vanishing of the outer flow velocity. For this reason these values were omitted from Fig. 5. One can, however, redefine the friction coefficient using the belt speed $w$. This has been carried out in Ref. 17 where the limiting value of the so defined friction coefficient is shown to be $-0.888$. Since this value is negative, it is clear that in decelerating flow the moving belt also always reduces the friction coefficient.

When the effect of the speed with which the belt is moving is considered, then Fig. 5 indicates that the rise of the friction coefficient downstream of the encounter plane is more rapid when this speed is slow than when it is fast. Thus for large $w/W$ and $\theta < 1$, the coefficient tends to be always negative regardless of the pressure gradient. This is in agreement with Eq. (18) which indicates that for $w/W > 1$ the ultimate values of the friction coefficient become larger and larger negative.

### C. Effect on the Momentum Thickness

As explained in Ref. 17, close to the encounter plane the variation of the momentum thickness is obtained by applying the momentum integral relation which gives

$$\frac{\delta_2}{\delta_2^0} = 1 - (1 + m)(1 + 3m)\,\frac{w}{W} \times \frac{\theta f_0''(0) + \frac{1}{2}\theta^2 f_1''(0)}{(1 + m)F''(0) - 2m(\eta^0 - F)_\infty} \tag{19a}$$

On the other hand, far from the encounter plane, the behavior of the momentum thickness may readily be obtained by substituting the appropriate functions into its definition and this results in

$$\frac{\delta_2}{\delta_2^0} = 1 + \frac{(1 + 3m)}{(1 + m)F''(0) - 2m(\eta^0 - F)_\infty}\left[\left(1 - \frac{\delta_1}{\delta_1^0}\right) \times (\eta^0 - F)_\infty - \frac{w}{W}\sum_{n=0}^{\infty} \zeta^{n+m} \sum_{k=0}^{n}\left((2I_{1,k} + \frac{w}{W}\,\zeta^m I_{2,k})a_{n-k}\right)\right] \tag{19b}$$

where the ratio of the displacement thickness (for $m \geq 0$) is given by

$$\frac{\delta_1}{\delta_1^0} = 1 - \frac{w}{W}\,\frac{\zeta^m}{(\eta^0 - F)_\infty} \times \{(\eta - f_0) - \zeta[f_1 + \tfrac{1}{2}(\eta - f_0)] - \zeta^2[f_2 - \tfrac{1}{2}f_1 + \tfrac{1}{8}(\eta - f_0)]\}_\infty \tag{20}$$

and the coefficients $I_{1,n}$ and $I_{2,n}$ are computed from

$$I_{1,n} = \int_0^\infty \left(S_n' - \sum_{k=0}^{n} S_k' f_{n-k}'\right) d\eta \tag{21a}$$

$$\left.\begin{aligned} I_{2,0} &= \int_0^\infty (1 - f_0')^2 d\eta \\ I_{2,n\neq 0} &= \int_0^\infty \left[\left(\sum_{k=0}^{n} f_k' f_{n-k}'\right) - 2f_n'\right] d\eta \end{aligned}\right\} \tag{21b}$$

Equations (19a) and (19b) are plotted in Fig. 6, except that again, for previously given reasons, the values for $\theta \rightarrow 1$ and $m = -0.090429$ are not included.

As can be seen from Fig. 6, in all cases the moving belt causes a decrease of the momentum thickness. Close to the encounter plane this effect becomes greater the larger the belt speed and the more accelerative the external pressure gradient. Far from the encounter plane, the behavior of the momentum thickness again depends primarily on the nature of the external pressure gradient. Thus, the continuously accelerating flow eventually nullifies the effect of the moving belt so that $\delta_2 \rightarrow \delta_2^0$. On the other hand, the flow with constant external velocity leads to limiting values given by Eq. (18). This equation indicates that for belt speeds larger than the freestream velocity the momentum thickness is always

‡‡ For higher values of $w/W$ improved correlation formulas are given in Ref. 11.

negative. Moreover, as can be seen in Fig. 6, when the belt speed is fast (relative to the freestream velocity), very large negative values of the momentum thickness are obtainable. These negative values indicate that the boundary layer fluid has actually gained momentum above that existing in the free stream. Therefore, since the momentum thickness may be shown to be related to the over-all frictional force acting on the body containing the moving belt, it is obvious that for negative values of the momentum thickness the moving belt is producing an over-all frictional thrust instead of drag and that an appreciable thrust force can be so generated.

## References


[1] Betz, A., "History of Boundary Layer Control in Germany," *Boundary Layer and Flow Control*, edited by G. V. Lachman, Vol. 1, Pergamon Press, New York, 1961, p. 1.

[2] Prandtl, L., *Essentials of Fluid Dynamics*, transl. W. M. Deans, Hafner, New York, 1952, p. 142.

[3] Goldstein, S., ed., *Modern Developments in Fluid Mechanics*, Vol. 1, Clarendon Press, Oxford, 1938, p. 79.

[4] Goodson, K. W., "Ground Effects on a Four-Propeller Tilt-Wing Configuration over a Fixed and a Moving Ground Plane," TN D-3938, May 1967, NASA.

[5] Calderon, A. A., "Rotating Cylinder Flaps for V/S.T.O.L. Aircraft," *Aircraft Engineering*, Vol. 36, No. 10, Oct. 1964, pp. 304–309; also see Brown, D. A., "Peruvians Study Rotating-Cylinder Flap," *Aviation Week and Space Technology*, Vol. 81, No. 23, Dec. 7, 1964, pp. 70–76.

[6] Carriere, P. and Eichelbrenner, E. A., "Theory of Flow Reattachment by a Tangential Jet Discharging Against a Strong Adverse Pressure Gradient," *Boundary Layer and Flow Control*, edited by G. V. Lachman, Vol. 1, Pergamon Press, New York, 1961, pp. 209–231.

[7] Adams, E. W. and Berry, B. K., "The Stationary Laminar Velocity Boundary Layer with Constant Fluid Properties and Arbitrary Distributions of Pressure and Mass Transfer," TN D-2642, Feb. 1965, NASA.

[8] Moore, F. K., "On the Separation of the Unsteady Laminar Boundary Layer," *Proceedings of the International Union of Theoretical and Applied Mechanics*, edited by H. Görtler, Springer Verlag, Berlin, 1958, pp. 296–311.

[9] Mirels, H., "Laminar Boundary Layer Behind Shock Advancing into Stationary Fluid," TN 3401, March 1955, NACA.

[10] Mirels, H., "Boundary Layer Behind Shock or Thin Expansion Wave Moving into Stationary Fluid," TN 3712, 1956, NACA.

[11] Mirels, H., "Correlation Formulas for Laminar Shock Tube Boundary Layer," *The Physics of Fluids*, Vol. 9, No. 7, July 1966, pp. 1265–1272.

[12] Sakiadis, B. C., "Boundary-Layer Behavior on Continuous Solid Surfaces: II. The Boundary Layer on a Continuous Flat Surface," *AIChE Journal*, Vol. 7, No. 2, June 1961, pp. 221–225.

[13] Tsou, F. K., Sparrow, E. M., and Goldstein, R. J., "Flow and Heat Transfer in the Boundary Layer on a Continuous Moving Surface," *International Journal of Heat and Mass Transfer*, Vol. 10, 1967, pp. 219–235.

[14] Görtler, H., "A New Series for the Calculation of Steady Laminar Boundary-Layer Flows," *Journal of Mathematics and Mechanics*, Vol. 6, No. 1, 1957, pp. 1–66.

[15] Falkner, V. M. and Skan, S. W., "Some Approximate Solutions of the Boundary Layer Equations," Reports Memo 1314, 1930, Aero Research Council, London.

[16] Hartree, D. R., "On an Equation Occurring in Falkner and Skan's Approximate Treatment of the Equations of the Boundary Layer," *Proceedings of the Cambridge Philosophical Society*, Vol. 33, Pt. 2, 1937, pp. 223–239.

[17] Mager, A., "Sudden Acceleration of a Laminar Boundary Layer by a Moving Belt," TR-0158(3305)-2, July 1968, Aerospace Corp., El Segundo, Calif.

[18] Lagestrom, P. A., "Laminar Flow Theory," Theory of Laminar Flows, *High Stream Aerodynamics and Jet Propulsion*, edited by F. K. Moore, Vol. 4, Princeton University Press, Princeton, N. J., 1964, pp. 121–124.

[19] Lock, R. C., "The Velocity Distribution in the Laminar Boundary Layer Between Parallel Streams," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 4, Pt. 1, 1951, pp. 42–63.